

# THE COLORING OF THE REGULAR GRAPH OF IDEALS <sup>\*†</sup>

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**ABSTRACT.** The regular graph of ideals of the commutative ring  $R$ , denoted by  $\Gamma_{reg}(R)$ , is a graph whose vertex set is the set of all non-trivial ideals of  $R$  and two distinct vertices  $I$  and  $J$  are adjacent if and only if either  $I$  contains a  $J$ -regular element or  $J$  contains an  $I$ -regular element. In this paper, it is shown that for every Artinian ring  $R$ , the edge chromatic number of  $\Gamma_{reg}(R)$  equals its maximum degree. Then a formula for the clique number of  $\Gamma_{reg}(R)$  is given. Also, it is proved that for every reduced ring  $R$  with  $n(\geq 3)$  minimal prime ideals, the edge chromatic number of  $\Gamma_{reg}(R)$  is  $2^{n-1} - 2$ . Moreover, we show that both of the clique number and vertex chromatic number of  $\Gamma_{reg}(R)$  are  $n - 1$ , for every reduced ring  $R$  with  $n$  minimal prime ideals.

## 1. Introduction

We begin with recalling some notations on graphs. Let  $\Gamma$  be a digraph. We denote the vertex set of  $\Gamma$ , by  $V(\Gamma)$ . Also, we distinguish the *out-degree*  $d_{\Gamma}^{+}(v)$ , the number of arcs leaving a vertex  $v$ , and the *in-degree*  $d_{\Gamma}^{-}(v)$ , the number of arcs entering a vertex  $v$ . If the graph is oriented, the degree  $d_{\Gamma}(v)$  of a vertex  $v$  is equal to the sum of its out- and in-degrees. Let  $G$  be a simple graph with the vertex set  $V(G)$  and  $A \subseteq V(G)$ . We denote by  $G[A]$  the subgraph of  $G$  induced by  $A$ . If  $|V(G)| = \mu$ , for some cardinal number  $\mu$ , then the complete graph and its complement are denoted by  $K_{\mu}$  and  $\overline{K_{\mu}}$ , respectively. The degree of a vertex  $x$  of  $G$  is denoted by  $d(x)$  and the maximum degree of vertices of  $G$  is denoted by  $\Delta(G)$ . A complete bipartite graph with parts of sizes  $\mu$  and  $\nu$  is denoted by  $K_{\mu,\nu}$ . Moreover, if either  $\mu = 1$  or  $\nu = 1$ , then the complete bipartite graph is said to be a *star graph*. Let  $G_1$  and  $G_2$  be two arbitrary graphs. By  $G_1 + G_2$  and  $G_1 \vee G_2$ , we mean the *disjoint union* of  $G_1$  and  $G_2$  and *join* of two graphs  $G_1$  and  $G_2$ , respectively. For a graph  $G$ , the *clique number* of  $G$ , and the *vertex (edge) chromatic number* of  $G$  are denoted by  $\omega(G)$ , and  $\chi(G)$  ( $\chi'(G)$ ), respectively. For more details about the used terminology of graphs, see [17].

Throughout this paper,  $R$  is assumed to be a non-domain commutative ring with identity. An element  $r \in R$  is called  *$R$ -regular* if  $r \notin Z(R)$ , where  $Z(R)$  denotes the set of all zero-divisors of  $R$ . By  $\mathbb{I}(R)$  ( $\mathbb{I}(R)^*$ ),  $Max(R)$  and  $Min(R)$  we denote the set of all proper (non-trivial) ideals of  $R$ , the set of all maximal ideals of  $R$  and the set of all minimal prime ideals of  $R$ , respectively.

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The ring  $R$  is said to be *reduced*, if it has no non-zero nilpotent element. For every ideal  $I$  of  $R$ , the *annihilator of  $I$*  is denoted by  $\text{Ann}(I)$ . A subset  $S$  of a commutative ring  $R$  is called a *multiplicative closed subset* (m.c.s) of  $R$  if  $1 \in S$  and  $x, y \in S$  implies that  $xy \in S$ . If  $S$  is an m.c.s of  $R$  and  $M$  is an  $R$ -module, then we denote by  $R_S$  and  $M_S$ , the ring of fractions of  $R$  and the module of fractions of  $M$  with respect to  $S$ , respectively. If  $\mathfrak{p}$  is a prime ideal of  $R$  and  $S = R \setminus \mathfrak{p}$ , we use the notation  $M_{\mathfrak{p}}$ , for the localization of  $M$  at  $\mathfrak{p}$ . By  $T(R)$ , we mean the *total ring* of  $R$  that is the ring of fractions, where  $S = R \setminus Z(R)$ .

As we know, most properties of a ring are closely tied to the behavior of its ideals, so it is useful to study graphs or digraphs, associated to the ideals of a ring or associated to modules. To see an instance of these graphs, the reader is referred to [1, 2, 4, 6, 9, 12, 13, 15, 16]. The *regular digraph of ideals* of a ring  $R$ , denoted by  $\overrightarrow{\Gamma_{\text{reg}}}(R)$ , is a digraph whose vertex set is the set of all non-trivial ideals of  $R$  and for every two distinct vertices  $I$  and  $J$ , there is an arc from  $I$  to  $J$  if and only if  $I$  contains a  $J$ -regular element. The underlying graph of  $\overrightarrow{\Gamma_{\text{reg}}}(R)$  is denoted by  $\Gamma_{\text{reg}}(R)$ . The regular digraph (graph) of ideals, first was introduced by Nikmehr and Shaveisi in [12]. Then in [3], Afkhami, Karimi and Khashayarmanesh followed the study of this graph. In this paper, the coloring of the regular graph of ideals is studied. In Section 2, it is shown that  $\chi'(\Gamma_{\text{reg}}(R)) = \Delta(\Gamma_{\text{reg}}(R))$ , where  $R$  is an Artinian ring. In Section 3, it is shown that  $\chi(\Gamma_{\text{reg}}(R)) = \omega(\Gamma_{\text{reg}}(R)) = 2|\text{Max}(R)| - f(R) - 1$ , where  $R$  is an Artinian ring and  $f(R)$  denotes the number of fields, appeared in the decomposition of  $R$  to direct product of local rings. Section 4 is devoted to the case that  $R$  is a reduced ring. For example, for every reduced ring  $R$  with  $|\text{Min}(R)| = n \geq 3$ , we obtain that  $\chi(\Gamma_{\text{reg}}(R)) = \omega(\Gamma_{\text{reg}}(R)) = n - 1$  and  $\chi'(\Gamma_{\text{reg}}(R)) = 2^{n-1} - 2$ .

## 2. The Edge Chromatic Number

In this section, we study the edge coloring of the regular graph of ideals of an Artinian ring. Before this, we need the following lemma from [7].

**Lemma 1.** [7, Corollary 5.4] *Let  $G$  be a simple graph. Suppose that for every vertex  $u$  of maximum degree, there exists an edge  $\{u, v\}$  such that  $\Delta(G) - d(v) + 2$  is more than the number of vertices with maximum degree in  $G$ . Then  $\chi'(G) = \Delta(G)$ .*

**Remark 2.** Let  $R_1, \dots, R_n$  be rings,  $R \cong R_1 \times \dots \times R_n$  and  $I = I_1 \times \dots \times I_n$  and  $J = J_1 \times \dots \times J_n$  be two distinct vertices of  $\Gamma_{\text{reg}}(R)$ . Then

- (i)  $I$  contains a  $J$ -regular element if and only if for every  $i$ , either  $I_i$  contains a  $J_i$ -regular element or  $J_i = (0)$ .
- (ii) Assume that every  $R_i$  is an Artinian local ring. Then (i) and [12, Theorem 2.1] imply that if  $I$  contains a  $J$ -regular element, then  $J$  contains no  $I$ -regular element.

**Theorem 3.** *If  $R$  is an Artinian ring, then  $\chi'(\Gamma_{\text{reg}}(R)) = \Delta(\Gamma_{\text{reg}}(R))$ .*

**Proof.** Let  $R$  be an Artinian ring. Then by [5, Theorem 8.7], there exists a positive integer  $n$  such that  $R \cong R_1 \times \cdots \times R_n$ , where every  $R_i$  is an Artinian local ring. If  $R$  contains infinitely many ideals, then with no loss of generality, we can assume that  $\mathbb{I}(R_1)$  is an infinite set. Since  $(0) \times R_2 \times \cdots \times R_n$  is adjacent to  $I_1 \times (0)$ , for every non-zero ideal  $I_1$  of  $R_1$ , we deduce that,  $\chi'(\Gamma_{reg}(R)) = \Delta(\Gamma_{reg}(R)) = \infty$ . Therefore, one can suppose that  $|\mathbb{I}(R)^*| < \infty$ . If  $R$  is a local ring, then by [12, Theorem 2.1],  $\Gamma_{reg}(R) \cong \overline{K_{|\mathbb{I}(R)^*|}}$  and hence  $\chi'(\Gamma_{reg}(R)) = \Delta(\Gamma_{reg}(R)) = 0$ . For the non-local case, we continue the proof in the following three cases:

Case 1.  $R$  is a reduced ring. Since  $R$  is Artinian, we conclude that  $R \cong F_1 \times \cdots \times F_n$ , where every  $F_i$  is a field. If  $n \leq 5$ , then it is not hard to check that  $\chi'(\Gamma_{reg}(R)) = \Delta(\Gamma_{reg}(R)) = 2^{n-1} - 2$ . Thus we can suppose that  $n \geq 6$ . Now, let  $I = F_1 \times \cdots \times F_r \times (0) \times \cdots \times (0)$  be a vertex of  $\Gamma_{reg}(R)$ , where  $1 \leq r \leq n-1$ . Then we have:

$$d(I) = d^+(I) + d^-(I) = 2^r - 2 + 2^{n-r} - 2 = 2^r + 2^{n-r} - 4.$$

Therefore, a vertex  $I = I_1 \times \cdots \times I_n$  of  $\Gamma_{reg}(R)$  has maximum degree if and only if either there exists exactly one  $j$ ,  $1 \leq j \leq n$  such that  $I_j = F_j$  or there exists exactly one  $j$ ,  $1 \leq j \leq n$  such that  $I_j = (0)$ . So, the number of vertices with maximum degree is  $2n$ . Now, let  $u$  be a vertex with maximum degree. Then with no loss of generality, we can suppose that either  $u = F_1 \times (0) \times \cdots \times (0)$  or  $u = (0) \times F_1 \times \cdots \times F_n$ . Suppose that  $u = F_1 \times (0) \times \cdots \times (0)$  and consider the vertex  $v = F_1 \times \cdots \times F_{\lfloor \frac{n}{2} \rfloor} \times (0) \times \cdots \times (0)$ . Clearly,  $d(u) = \Delta(\Gamma_{reg}(R)) = 2^{n-1} - 2$ ,  $d(v) = 2^{\lfloor \frac{n}{2} \rfloor} + 2^{n-\lfloor \frac{n}{2} \rfloor} - 4$  and  $u$  is adjacent to  $v$ . Since  $n \geq 6$ , we deduce that

$$\Delta(\Gamma_{reg}(R)) - d(v) + 2 = 2^{n-1} - 2^{n-\lfloor \frac{n}{2} \rfloor} - 2^{\lfloor \frac{n}{2} \rfloor} + 4 > 2n.$$

If  $u = (0) \times F_1 \times \cdots \times F_n$ , then a similar proof to that of above shows that  $\Delta(\Gamma_{reg}(R)) - d(v) + 2 > 2n$ . Thus Lemma 1 implies that  $\chi'(\Gamma_{reg}(R)) = \Delta(\Gamma_{reg}(R))$ .

Case 2.  $R$  is a non-reduced ring and  $|\text{Max}(R)| = 2$ . In this case,  $R \cong R_1 \times R_2$ , where  $R_1$  and  $R_2$  are Artinian local rings. Let  $|V(\Gamma_{reg}(R_1))| = \mu$  and  $|V(\Gamma_{reg}(R_2))| = \nu$ ,

$$A = V(\Gamma_{reg}(R_1)) \times V(\Gamma_{reg}(R_2)),$$

$$B_1 = V(\Gamma_{reg}(R_1)) \times \{(0)\}, B_2 = \{R_1\} \times V(\Gamma_{reg}(R_2)), B_3 = \{R_1 \times (0)\},$$

$$C_1 = V(\Gamma_{reg}(R_1)) \times \{R_2\}, C_2 = \{(0)\} \times V(\Gamma_{reg}(R_2)), C_3 = \{(0) \times R_2\},$$

$$B = B_1 \cup B_2 \cup B_3 \text{ and } C = C_1 \cup C_2 \cup C_3.$$

Then we have:

$$\Gamma_{reg}(R) \cong \Gamma_{reg}(R)[A] + \Gamma_{reg}(R)[B] + \Gamma_{reg}(R)[C] \cong \overline{K_{\mu\nu}} + (\overline{K_\mu} \vee K_{1,\nu}) + (\overline{K_\mu} \vee K_{1,\nu}).$$

So,  $\chi'(\Gamma_{reg}(R)) = \mu + \nu = \Delta(\Gamma_{reg}(R))$ , as desired.

Case 3.  $R$  is a non-reduced ring and  $|\text{Max}(R)| = n \geq 3$ . Let  $I = I_1 \times \cdots \times I_n$  be a non-trivial

ideal of  $R$  and define the following sets and numbers:

$$\begin{aligned}\Delta_I &= \{k \mid 1 \leq k \leq n \text{ and } I_k = R_k\}; \\ \Upsilon_I &= \{k \mid 1 \leq k \leq n \text{ and } I_k = (0)\}; \\ \Lambda_I &= \{k \mid 1 \leq k \leq n, \text{ and } I_k \text{ is a non-trivial ideal of } R_k\}; \\ t_i &= |\mathbb{I}(R_i)|; \quad (1 \leq i \leq n); \\ T_i &= \{j \mid 1 \leq j \leq n \text{ and } |\mathbb{I}(R_j)| = t_i\}; \quad s_i = |T_i| \quad (1 \leq i \leq n).\end{aligned}$$

With no loss of generality, we can assume that  $t_1 \geq \dots \geq t_n$ . Now, let us compute the degree of every vertex of  $\Gamma_{reg}(R)$ . By Remark 2, there is an arc from  $I$  to  $J$  in  $\overrightarrow{\Gamma_{reg}}(R)$  if and only if  $\Upsilon_J \supseteq \Upsilon_I \cup \Lambda_I$ . So, the out-degree of  $I$  in  $\overrightarrow{\Gamma_{reg}}(R)$  equals:

$$d^+(I) = \begin{cases} 0; & \Delta_I = \emptyset \\ \prod_{k \in \Delta_I} (t_k + 1) - 2; & \Delta_I \neq \emptyset \text{ and } \Lambda_I = \emptyset \\ \prod_{k \in \Delta_I} (t_k + 1) - 1; & \Delta_I \neq \emptyset \text{ and } \Lambda_I \neq \emptyset \end{cases}$$

Also, Remark 2 implies that there is an arc from  $J$  to  $I$  in  $\overrightarrow{\Gamma_{reg}}(R)$  if and only if  $\Delta_J \supseteq \Delta_I \cup \Lambda_I$ . Thus the in-degree of  $I$  in  $\overrightarrow{\Gamma_{reg}}(R)$  equals:

$$d^-(I) = \begin{cases} 0; & \Upsilon_I = \emptyset \\ \prod_{k \in \Upsilon_I} (t_k + 1) - 2; & \Upsilon_I \neq \emptyset \text{ and } \Lambda_I = \emptyset \\ \prod_{k \in \Upsilon_I} (t_k + 1) - 1; & \Upsilon_I \neq \emptyset \text{ and } \Lambda_I \neq \emptyset. \end{cases}$$

Therefore,

$$d(I) = \begin{cases} 0; & \Lambda_I = \{1, \dots, n\} \\ \prod_{k \in \Delta_I} (t_k + 1) + \prod_{k \in \Upsilon_I} (t_k + 1) - 4; & \Lambda_I = \emptyset \\ \prod_{k \in \Upsilon_I} (t_k + 1) - 1; & \Lambda_I \neq \emptyset, \Upsilon_I \neq \emptyset \text{ and } \Delta_I = \emptyset \\ \prod_{k \in \Delta_I} (t_k + 1) - 1; & \Lambda_I \neq \emptyset, \Upsilon_I = \emptyset \text{ and } \Delta_I \neq \emptyset \\ \prod_{k \in \Delta_I} (t_k + 1) + \prod_{k \in \Upsilon_I} (t_k + 1) - 2; & \Lambda_I \neq \emptyset, \Upsilon_I \neq \emptyset \text{ and } \Delta_I \neq \emptyset. \end{cases}$$

Now, we consider the following two subcases:

Subcase 1.  $R$  contains no field as its direct summand. From the above argument, we conclude that  $I$  has maximum degree if and only if either  $\Delta_I = \{1, \dots, n\} \setminus \{j\}$  and  $\Lambda_I = \{j\}$  or  $\Upsilon_I = \{1, \dots, n\} \setminus \{j\}$  and  $\Lambda_I = \{j\}$ , for some  $j \in T_n$ . Let  $u$  be a vertex with maximum degree in  $\Gamma_{reg}(R)$ . Then with no loss of generality, we can suppose that either  $u = R_1 \times \dots \times R_{n-1} \times I_n$  or  $u = (0) \times \dots \times (0) \times J_n$ , where  $I_n$  and  $J_n$  are non-trivial ideals of  $R_n$ . First suppose that  $u = R_1 \times \dots \times R_{n-1} \times I_n$ , where  $I_n$  is non-trivial ideal of  $R_n$ . Consider the vertex  $v = \mathfrak{m}_1 \times \dots \times \mathfrak{m}_{n-1} \times (0)$ , where  $\mathfrak{m}_i$  is the maximal ideal of  $R_i$ , for every  $1 \leq i \leq n$ . By Remark 2, a vertex  $J$  in  $\Gamma_{reg}(R)$  is

adjacent to  $v$  if and only if  $J = R_1 \times \cdots \times R_{n-1} \times J_n$ , for some proper ideal  $J_n$  of  $R_n$ . Therefore, the following statements are true:

- (a)  $u$  and  $v$  are two adjacent vertices in  $\Gamma_{reg}(R)$  and  $d(v) = t_n$ .
- (b)  $\Delta(\Gamma_{reg}(R)) = d(u) = \prod_{k=1}^{n-1} (t_k + 1) - 1$ .
- (c) The number of vertices with maximum degree in  $\Gamma_{reg}(R)$  is  $2s_n(t_n - 1)$ .

Since  $n \geq 3$  and  $t_i \geq 2$ , for every  $i \geq 1$ , from the above statements, we have:

$$\begin{aligned} \Delta(\Gamma_{reg}(R)) - d(v) + 2 - 2s_n(t_n - 1) &= \prod_{k=1}^{n-1} (t_k + 1) - (2s_n + 1)(t_n - 1) \\ &\geq 3^{n-2}(t_n + 1) - (2n + 1)(t_n - 1) \\ &= (3^{n-2} - 2n - 1) + 2(3^{n-2}) > 0 \end{aligned}$$

Hence  $\Delta(\Gamma_{reg}(R)) - d(v) + 2$  is more than the number of vertices with maximum degree in  $\Gamma_{reg}(R)$ . Thus by Lemma 1,  $\chi'(\Gamma_{reg}(R)) = \Delta(\Gamma_{reg}(R))$ . Now, suppose that  $u = (0) \times \cdots \times (0) \times J_n$ , for some non-trivial ideal  $J_n$  of  $R_n$ . In this case, consider the vertex  $v = \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_{n-1} \times R_n$ , where  $\mathfrak{m}_i$  is the maximal ideal of  $R_i$ , for every  $1 \leq i \leq n$ . Then a similar argument to that of above shows that  $u$  and  $v$  are adjacent and  $\Delta(\Gamma_{reg}(R)) - d(v) + 2$  is more than the number of vertices with maximum degree in  $\Gamma_{reg}(R)$ . So, Lemma 1 implies that  $\chi'(\Gamma_{reg}(R)) = \Delta(\Gamma_{reg}(R))$ .

Subcase 2.  $R$  contains a field as its direct summand. In this case,  $R_n$  is a field. From the argument before subcase 1, we conclude that  $I$  has maximum degree if and only if either  $\Delta_I = \{1, \dots, n\} \setminus \{j\}$  and  $\Upsilon_I = \{j\}$  or  $\Upsilon_I = \{1, \dots, n\} \setminus \{j\}$  and  $\Delta_I = \{j\}$ , for some  $j \in T_n$ . So, if  $u$  is a vertex with maximum degree in  $\Gamma_{reg}(R)$ , then with no loss of generality, we can suppose that either  $u = R_1 \times \cdots \times R_{n-1} \times (0)$  or  $u = (0) \times \cdots \times (0) \times R_n$ . First suppose that  $u = R_1 \times \cdots \times R_{n-1} \times (0)$ . Consider the vertex  $v = \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_{n-k} \times (0) \times \cdots \times (0)$ , where  $k$  is the number of fields, appearing in the decomposition of  $R$  to local rings. Then it is clear that a vertex  $J = J_1 \times \cdots \times J_n$  is adjacent to  $v$  if and only if  $J_i = R_i$ , for every  $1 \leq i \leq n - k$ . Therefore, the following statements are true:

- (a')  $u$  and  $v$  are two adjacent vertices in  $\Gamma_{reg}(R)$  and  $d(v) = 2^k$ .
- (b')  $\Delta(\Gamma_{reg}(R)) = d(u) = \prod_{k=1}^{n-1} (t_k + 1) - 2$ .
- (c') The number of vertices with maximum degree in  $\Gamma_{reg}(R)$  is  $2k$ .

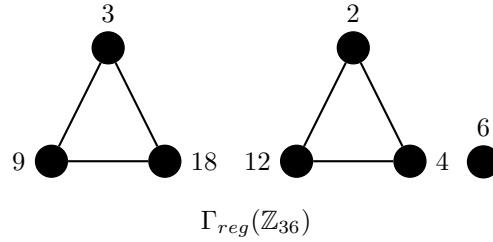
Thus  $\Delta(\Gamma_{reg}(R)) - d(v) + 2 = \prod_{k=1}^{n-1} (t_k + 1) - 2^k$ . Since  $R$  is not reduced and  $n \geq 3$ , we deduce that this number is greater than the number of vertices with maximum degree in  $\Gamma_{reg}(R)$  and hence Lemma 1 implies that  $\chi'(\Gamma_{reg}(R)) = \Delta(\Gamma_{reg}(R))$ . Now, assume that  $u = (0) \times \cdots \times (0) \times R_n$  and consider the vertex  $v = \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_{n-k} \times R_{k+1} \times \cdots \times R_n$ . Then a similar proof to that of above shows that  $u$  and  $v$  are adjacent and we have:

$$\Delta(\Gamma_{reg}(R)) - d(v) + 2 = \prod_{k=1}^{n-1} (t_k + 1) - 2^k + 1 \geq 3^{n-1} - 2^k + 1,$$

which is greater than the number of vertices with maximum degree in  $\Gamma_{reg}(R)$ . Thus by Lemma 1,  $\chi'(\Gamma_{reg}(R)) = \Delta(\Gamma_{reg}(R))$  and so the proof is complete.  $\square$

### 3. A Formula for the Clique Number in Artinian Rings

Let  $R = \mathbb{Z}_{36} \cong \mathbb{Z}_4 \times \mathbb{Z}_9$ . Then it is clear that  $R$  is an Artinian ring with two maximal ideals. On the other hand, we know that  $C = \{\mathbb{Z}_4 \times (0), \mathbb{Z}_4 \times (3), (2) \times (0)\}$  is a clique in  $\Gamma_{reg}(R)$  and so  $\omega(\Gamma_{reg}(R)) \geq 3 > |\text{Max}(R)|$ ; therefore, this implies that the upper bound for  $\omega(\Gamma_{reg}(R))$  in [12, Theorem 2.1] is incorrect. The regular digraph of  $\mathbb{Z}_{36}$  is seen in the following figure:



In this section, we give a correct upper bound for  $\omega(\Gamma_{reg}(R))$ , when  $R$  is an Artinian ring. In fact, it is shown that for every Artinian ring  $R$ ,  $|\text{Max}(R)| - 1 \leq \omega(\Gamma_{reg}(R)) \leq 2|\text{Max}(R)| - 1$  and the lower bound occurs if and only if  $R$  is a reduced ring. If  $R$  is an Artinian local ring which is not a field, then by [12, Theorem 2.1],  $\omega(\Gamma_{reg}(R)) = 1$ . Also, it is clear that for every field  $F$ ,  $\omega(\Gamma_{reg}(F)) = 0$ .

**Lemma 4.** *Let  $S$  be an Artinian ring and  $T$  be an Artinian local ring. If  $R \cong S \times T$ , then*

$$\omega(\Gamma_{reg}(R)) = \begin{cases} \omega(\Gamma_{reg}(S)) + 1; & T \text{ is a field} \\ \omega(\Gamma_{reg}(S)) + 2; & T \text{ is not a field.} \end{cases}$$

**Proof.** First note that for every clique  $C$  of  $\Gamma_{reg}(S)$ ,  $C \times \{T\}$  is a clique of  $\Gamma_{reg}(R)$ . Also, for any clique  $C' = \{I_i \times J_i\}_{i \in A}$  of  $\Gamma_{reg}(R)$ , from Remark 2 and [12, Theorem 2.1], we deduce that  $\{J_i \mid I_i \times J_i \in C'\}_{i \in A}$  contains at most one nontrivial ideal. Therefore,  $\omega(\Gamma_{reg}(S))$  is infinite if and only if  $\omega(\Gamma_{reg}(R))$  is infinite. Now, assume that  $\omega(\Gamma_{reg}(S))$  is finite and  $C$  is a clique of  $\Gamma_{reg}(S)$  with  $|C| = \omega(\Gamma_{reg}(S))$ . If  $T$  is a field, then  $C \times \{T\} \cup \{(0) \times T\}$  is a clique of  $\Gamma_{reg}(S)$ . Also, if  $T$  is not a field, then for every nontrivial ideal  $J$  of  $T$ ,  $C \times \{T\} \cup \{(0) \times J, (0) \times T\}$  is a clique of  $\Gamma_{reg}(R)$ . Therefore,

$$\omega(\Gamma_{reg}(R)) \geq \begin{cases} \omega(\Gamma_{reg}(S)) + 1; & T \text{ is a field} \\ \omega(\Gamma_{reg}(S)) + 2; & T \text{ is not a field.} \end{cases}$$

Next, we prove the inverse inequality. To see this, let  $C' = \{I_i \times J_i \mid 1 \leq i \leq t\}$  be a maximal clique of  $\Gamma_{reg}(R)$ . Setting

$$C_1 = \{I_i \times J_i \in C' \mid J_i \text{ is a nontrivial ideal of } T\},$$

we deduce that there are sets  $C_2, C_3$  such that

$$C' = C_1 \cup (C_2 \times \{(0)\}) \cup (C_3 \times \{T\}).$$

Hence

$$|C'| = |C_1| + |C_2| + |C_3|. \quad (1)$$

From [12, Theorem 2.1] and Remark 2, it follows that  $|C_1| \leq 1$ ; moreover, if  $T$  is a field, then  $|C_1| = 0$ . Now, we follow the proof in the following two cases:

Case 1. Either  $C_2 = \emptyset$  or  $C_3 = \emptyset$ . If  $C_2 = \emptyset$  (resp.  $C_3 = \emptyset$ ), then by Remark 2,  $C_3 \setminus \{(0)\}$  (resp.  $C_2 \setminus \{T\}$ ) is a clique of  $\Gamma_{reg}(S)$ . This implies that  $|C_2| + |C_3| \leq \omega(\Gamma_{reg}(S)) + 1$ . Thus by (1), we have:

$$\omega(\Gamma_{reg}(R)) = |C'| \leq \begin{cases} \omega(\Gamma_{reg}(S)) + 1; & T \text{ is a field} \\ \omega(\Gamma_{reg}(S)) + 2; & T \text{ is not a field.} \end{cases}$$

Case 2.  $C_2 \neq \emptyset$  and  $C_3 \neq \emptyset$ . In this case, one can easily check that  $C_2$  and  $C_3$  contain only nontrivial ideals. Also, it follows from Remark 2 that  $C_2 \cup C_3$  is a clique of  $\Gamma_{reg}(S)$ , and this implies that  $|C_2 \cup C_3| \leq \omega(\Gamma_{reg}(S))$ . We claim that  $|C_2 \cap C_3| \leq 1$ . Suppose to the contrary,  $I_1, J_1 \in C_2 \cap C_3$ . Then it is clear that  $I_1, J_1$  are nontrivial ideals of  $S$ , and  $\{I_1 \times (0), I_1 \times T, J_1 \times (0), J_1 \times T\} \subseteq C'$ . Thus  $\overrightarrow{\Gamma_{reg}}(R)$  contains the arcs  $I_2 \times T \longrightarrow I_1 \times (0)$  and  $I_1 \times T \longrightarrow I_2 \times (0)$ . Hence  $I_1$  contains an  $I_2$ -regular element and  $I_2$  contains an  $I_1$ -regular element, and this contradicts Remark 2(ii). So the claim is proved and hence,

$$|C_2| + |C_3| = |C_2 \cup C_3| + |C_2 \cap C_3| \leq \omega(\Gamma_{reg}(S)) + 1.$$

Thus again by (1), we have:

$$\omega(\Gamma_{reg}(R)) = |C'| \leq \begin{cases} \omega(\Gamma_{reg}(S)) + 1; & T \text{ is a field} \\ \omega(\Gamma_{reg}(S)) + 2; & T \text{ is not a field.} \end{cases}$$

Therefore, in any case, the assertion follows.  $\square$

For any Artinian ring  $R$ , by  $f(R)$ , we denote the number of fields, appeared in the decomposition of  $R$  to direct product of local rings.

**Proposition 5.** *For any Artinian ring  $R$ ,  $\omega(\Gamma_{reg}(R)) = 2|\text{Max}(R)| - f(R) - 1$ .*

**Proof.** If  $R$  is a field, then there is nothing to prove. So, assume that  $R$  is an Artinian ring which is not a field. Then [5, Theorem 8.7] implies that  $R \cong R_1 \times R_2 \times \cdots \times R_n$ , where  $n = |\text{Max}(R)|$

and every  $R_i$  is an Artinian local ring. We prove the assertion, by induction on  $n$ . If  $n = 1$ , then the assertion follows from [12, Theorem 2.1]. Thus we can assume that  $n \geq 2$ . Now, setting  $S = R_1 \times R_2 \times \cdots \times R_{n-1}$ , we follow the proof in the following two cases:

Case 1.  $R_n$  is a field. In this case, the induction hypothesis implies that

$$\omega(\Gamma_{reg}(R')) = 2|\text{Max}(R')| - f(R') - 1 = 2(n-1) - (f(R) - 1) - 1 = 2n - f(R) - 2;$$

Thus by Lemma 4, we have:

$$\omega(\Gamma_{reg}(R)) = \omega(\Gamma_{reg}(R')) + 1 = 2n - f(R) - 1.$$

Case 2.  $R_n$  is not a field. In this case, the induction hypothesis implies that

$$\omega(\Gamma_{reg}(R')) = 2|\text{Max}(R')| - f(R') - 1 = 2(n-1) - f(R) - 1 = 2n - f(R) - 3;$$

Thus again by Lemma 4, we have:

$$\omega(\Gamma_{reg}(R)) = \omega(\Gamma_{reg}(R')) + 2 = 2n - f(R) - 1.$$

Therefore, in any case, the assertion follows.  $\square$

From [12, Theorem 2.3] and Proposition 5, we have the following corollary.

**Corollary 6.** *Let  $R$  be an Artinian ring. Then*

- (i)  $\omega(\Gamma_{reg}(R)) = \chi(\Gamma_{reg}(R))$ .
- (ii) *If  $R$  is reduced, then  $\omega(\Gamma_{reg}(R)) = |\text{Max}(R)| - 1$ .*

Now, we state the correct version of Theorem 2.2 from [12].

**Theorem 7.** *If  $R$  is an Artinian ring, then  $|\text{Max}(R)| - 1 \leq \omega(\Gamma_{reg}(R)) \leq 2|\text{Max}(R)| - 1$ . Moreover,  $\omega(\Gamma_{reg}(R)) = |\text{Max}(R)| - 1$  if and only if  $R$  is reduced.*

#### 4. The Case that $R$ is a Reduced ring

In this section the clique number, the vertex chromatic number and the edge chromatic number of  $\Gamma_{reg}(R)$  are determined, when  $R$  is a reduced ring. First, we recall the following interesting result, due to Eben Matlis.

**Proposition 8.** [11, Proposition 1.5] *Let  $R$  be a ring and  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$  be a finite set of distinct minimal prime ideals of  $R$ . Let  $S = R \setminus \bigcup_{i=1}^n \mathfrak{p}_i$ . Then  $R_S \cong R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_n}$ .*



**Theorem 9.** *Let  $R$  be a reduced ring,  $|\text{Min}(R)| = n \geq 3$  and  $\omega(\Gamma_{\text{reg}}(R)) < \infty$ . Then we have:*

- (i)  $\omega(\Gamma_{\text{reg}}(R)) = \chi(\Gamma_{\text{reg}}(R)) = \chi(\Gamma_{\text{reg}}(T(R))) = \omega(\Gamma_{\text{reg}}(T(R))) = n - 1$ .
- (ii)  $\chi'(\Gamma_{\text{reg}}(R)) = \chi'(\Gamma_{\text{reg}}(T(R))) = \begin{cases} 2^{n-1} - 2; & n \geq 3 \\ 0; & n = 2. \end{cases}$

**Proof.** Assume that  $\omega(\Gamma_{\text{reg}}(R)) < \infty$ . First we show that every element of  $R$  is either zero-divisor or unit. By contrary, suppose that  $x \in R$  is neither zero-divisor nor unit. Then it is not hard to check that  $\{(x^n)\}_{n \geq 1}$  is an infinite clique of  $\Gamma_{\text{reg}}(R)$ , a contradiction. Suppose that  $\text{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ , for some positive integer  $n$ . If  $S = R \setminus \bigcup_{i=1}^n \mathfrak{p}_i$ , then [10, Corollary 2.4] implies that  $T(R) = R_S$ . So by Proposition 8, we have  $T(R) \cong R_{\mathfrak{p}_1} \times \dots \times R_{\mathfrak{p}_n}$ . Since  $R$  is reduced, by [11, Proposition 1.1], Part (1), every  $R_{\mathfrak{p}_i}$  is a field. We claim that if  $I$  and  $J$  are two distinct vertices of  $\overrightarrow{\Gamma_{\text{reg}}}(R)$ , then  $I \longrightarrow J$  is an arc in  $\overrightarrow{\Gamma_{\text{reg}}}(R)$  if and only if  $I_S \longrightarrow J_S$  is an arc in  $\overrightarrow{\Gamma_{\text{reg}}}(R_S)$ . First suppose that  $I$  and  $J$  are two distinct non-trivial ideals of  $R$  and there is an arc from  $I$  to  $J$  in  $\overrightarrow{\Gamma_{\text{reg}}}(R)$ . Since  $S$  contains no zero-divisor, we deduce that  $I_S$  and  $J_S$  are two non-trivial ideals of  $R_S$ . We show that  $I_S \neq J_S$ . Suppose to the contrary,  $I_S = J_S$ . Then for every  $x \in I$ , there exists an element  $t \in S$  such that  $tx \in J$ . Since every element in  $S$  is a unit, we deduce that  $x \in J$ . So  $I \subseteq J$ . Similarly, one can show that  $J \subseteq I$ . Thus  $I = J$ , a contradiction. Therefore,  $I_S \neq J_S$ . Now, let  $x \in I$  be a  $J$ -regular element. Then one can easily show that  $\frac{x}{1} \in I_S$  is a  $J_S$ -regular element and so there is an arc from  $I_S$  to  $J_S$  in  $\overrightarrow{\Gamma_{\text{reg}}}(R_S)$ . Conversely, let  $\frac{x}{s} \in I_S$  be a  $J_S$ -regular element. Then we show that  $x \in I$  is a  $J$ -regular element. Suppose to the contrary,  $xy = 0$ , for some  $0 \neq y \in J$ . Then we deduce that  $\frac{x}{s} \cdot \frac{y}{1} = 0$ , a contradiction. So the claim is proved. Therefore, the graphs  $\Gamma_{\text{reg}}(R)$  and  $\Gamma_{\text{reg}}(T(R))$  are isomorphic. Now, since  $T(R)$  is the direct product of  $n$  fields, (i) follows from Proposition 5. Next, we prove (ii). By Theorem 3, we have  $\chi'(\Gamma_{\text{reg}}(T(R))) = \Delta(\Gamma_{\text{reg}}(T(R)))$ . Note that if  $n = 2$ , then  $T(R)$  is a direct product of two fields and hence  $\Gamma_{\text{reg}}(T(R))$  contains no edge. As we saw in the proof of Theorem 3,  $\Delta(\Gamma_{\text{reg}}(T(R))) = 2^{n-1} - 2$ , for every  $n \geq 3$ . Therefore,  $\chi'(\Gamma_{\text{reg}}(R)) = 2^{n-1} - 2$ , and the proof is complete.  $\square$

The following corollary is an immediate consequence of Theorem 9.

**Corollary 10.** *Let  $R$  be a reduced ring with finitely many minimal prime ideals such that  $\omega(\Gamma_{\text{reg}}(R)) < \infty$ . Then*

$$|\text{Min}(R)| = |\text{Max}(T(R))| = \chi(\Gamma_{\text{reg}}(T(R))) + 1 = \omega(\Gamma_{\text{reg}}(R)) + 1.$$

Finally, in the remaining of this paper, we see that the finiteness of the clique number and vertex chromatic number of the regular graph of ideals of  $R$  depends on those of localizations of  $R$  at maximal ideals. Before this, we need to recall the following lemma from [1].

**Lemma 11.** (See [1, Lemma 9]) *Let  $R$  be a ring,  $I$  and  $J$  be two non-trivial ideals of  $R$ . If for every  $\mathfrak{m} \in \text{Max}(R)$ ,  $I_{\mathfrak{m}} = J_{\mathfrak{m}}$ , then  $I = J$ .*

**Remark 12.** Let  $I$  and  $J$  be two distinct non-trivial ideals of  $R$  such that  $I \longrightarrow J$  is an arc of  $\overrightarrow{\Gamma_{reg}(R)}$ . Then from [8, Proposition 1.2.3], we deduce that  $\text{Hom}_R(\frac{R}{I}, J) = 0$ . Moreover, if  $R$  is a Noetherian ring, then  $\text{Hom}(\frac{R}{I}, J) = 0$  implies that  $I \longrightarrow J$  is an arc of  $\overrightarrow{\Gamma_{reg}(R)}$ .

**Theorem 13.** *Let  $R$  be a Noetherian ring with finitely many maximal ideals. If for every  $\mathfrak{m} \in \text{Max}(R)$ ,  $\omega(\Gamma_{reg}(R_{\mathfrak{m}}))$  is finite, then  $\omega(\Gamma_{reg}(R))$  is finite.*

**Proof.** Let  $\text{Max}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ . Suppose to the contrary,  $C = \{J_i\}_{i=1}^{\infty}$  is an infinite clique of  $\Gamma_{reg}(R)$ . Then by Remark 12, for every  $i$  and  $j$  with  $i \neq j$ , either  $\text{Hom}_R(\frac{R}{J_i}, J_j) = 0$  or  $\text{Hom}_R(\frac{R}{J_j}, J_i) = 0$ . Thus from [14, Lemma 4.87], we obtain that  $\text{Hom}_{R_{\mathfrak{m}_1}}(\frac{R_{\mathfrak{m}_1}}{(J_i)_{\mathfrak{m}_1}}, (J_j)_{\mathfrak{m}_1}) = 0$  or  $\text{Hom}_{R_{\mathfrak{m}_1}}(\frac{R_{\mathfrak{m}_1}}{(J_j)_{\mathfrak{m}_1}}, (J_i)_{\mathfrak{m}_1}) = 0$ , for every  $i$  and  $j$  with  $i \neq j$ . Since  $\omega(\Gamma_{reg}(R_{\mathfrak{m}_1})) < \infty$ , we deduce that there exists an infinite subset  $A_1 \subseteq \mathbb{N}$  such that for every  $i, j \in A_1$ ,  $(J_i)_{\mathfrak{m}_1} = (J_j)_{\mathfrak{m}_1}$ . Now, using  $\omega(\Gamma_{reg}(R_{\mathfrak{m}_2})) < \infty$ , we conclude that there exists an infinite subset  $A_2 \subseteq A_1$  such that for every  $i, j \in A_2$ ,  $(J_i)_{\mathfrak{m}_2} = (J_j)_{\mathfrak{m}_2}$ . By continuing this procedure one can see that there exists an infinite subset  $A_n \subseteq A_{n-1}$  such that for every  $i, j \in A_n$ ,  $(J_i)_{\mathfrak{m}_l} = (J_j)_{\mathfrak{m}_l}$ , for every  $l$ ,  $l = 1, \dots, n$ . Therefore, by Lemma 11, we get a contradiction.  $\square$

**Theorem 14.** *Let  $R$  be a ring with finitely many maximal ideals. If for every  $\mathfrak{m} \in \text{Max}(R)$ ,  $\chi(\Gamma_{reg}(R_{\mathfrak{m}}))$  is finite, then  $\chi(\Gamma_{reg}(R))$  is finite and moreover,*

$$\chi(\Gamma_{reg}(R)) \leq \prod_{\mathfrak{m} \in \text{Max}(R)} (\chi(\Gamma_{reg}(R_{\mathfrak{m}})) + 2) - 2.$$

**Proof.** Let  $\text{Max}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$  and  $f_i : V(\Gamma_{reg}(R_{\mathfrak{m}_i})) \longrightarrow \{1, \dots, \chi(\Gamma_{reg}(R_{\mathfrak{m}_i}))\}$  be a proper vertex coloring of  $\Gamma_{reg}(R_{\mathfrak{m}_i})$ , for every  $i$ ,  $1 \leq i \leq n$ . We define a function  $f$  on  $\mathbb{I}(R) \setminus \{R\}$  by  $f(I) = (g_1(I_{\mathfrak{m}_1}), \dots, g_n(I_{\mathfrak{m}_n}))$ , where

$$g_i(I_{\mathfrak{m}_i}) = \begin{cases} 0; & I_{\mathfrak{m}_i} = (0) \\ -1; & I_{\mathfrak{m}_i} = R_{\mathfrak{m}_i} \\ f_i(I_{\mathfrak{m}_i}); & \text{otherwise.} \end{cases}$$

Using Lemma 11, it is not hard to check that  $f$  is a proper vertex coloring of  $\Gamma_{reg}(R)$  and this completes the proof.  $\square$

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